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Matching QCD and HQET heavy–light currents at two loops and beyond

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Abstract Heavy–light QCD currents are matched with HQET currents at two loops and leading order in $1/m$. A single formula applies to all current matchings. As a by–product, a master formula for the two–loop anomalous dimension of the QCD current $\bar{q}\gamma^{[\mu_1}\ldots\gamma^{\mu_n]}q$ is obtained, yielding a new result for the tensor current. The dependence of matching coefficients on γ_5 prescriptions is elucidated. Ratios of QCD matrix elements are obtained, independently of the three–loop anomalous dimension of HQET currents. The two–loop coefficient in $f_{\mathrm{B}^*}/f_{\mathrm{B}} = 1 - 2\alpha_{\mathrm{s}}(m_b)/3\pi - K_b\alpha_{\mathrm{s}}^2/\pi^2 + \mathrm{O}(\alpha_{\mathrm{s}}^3, 1/m_b)$ is

$$K_b = \frac{83}{12} + \frac{4}{81}\pi^2 + \frac{2}{27}\pi^2\log 2 - \frac{1}{9}\zeta(3) - \frac{19}{54}N_l + \Delta_c = 6.37 + \Delta_c$$

with $N_l = 4$ light flavours, and a correction, $\Delta_c = 0.18 \pm 0.01$, that takes account of the non–zero ratio $m_c/m_b = 0.28 \pm 0.03$. Fastest apparent convergence would entail $\alpha_{\mathrm{s}}(\mu)$ at $\mu = 370$ MeV. "Naive non–abelianization" of large–N_l results, via $N_l \to N_l - \frac{33}{2}$, gives reasonable approximations to exact two–loop results. All–order results for anomalous dimensions and matching coefficients are obtained at large $\beta_0 = 11 - \frac{2}{3}N_l$. Consistent cancellation between infrared– and ultraviolet–renormalon ambiguities is demonstrated.

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1 Introduction

QCD problems with a single heavy quark staying approximately at rest are conveniently described by an effective field theory—HQET [1, 2] (see [3] for review and references). QCD operators are expanded, in powers of $1/m$, in terms of HQET operators, m being the on–shell heavy–quark mass. In this paper we study, at the two–loop level, the relation between currents in the full theory and the effective theory.

Specifically, we consider heavy–light bilinear currents, $J_0 = \bar{q}_0 \Gamma Q_0$, where Γ is a Dirac matrix and the subscript 0 denotes an unrenormalized quantity. The $\overline{\text{MS}}$–renormalized QCD current, $J(\mu) = Z_J^{-1}(\mu) J_0$, is expanded in HQET operators as

$$J(\mu) = C_\Gamma(\mu) \tilde{J}(\mu) + \frac{1}{m} \sum_i B_i(\mu) \tilde{O}_i(\mu) + \mathrm{O}\left(\frac{1}{m^2}\right), \tag{1.1}$$

where $\tilde{J}(\mu) = \tilde{Z}_J^{-1}(\mu) \tilde{J}_0$ is the corresponding renormalized HQET current, $\tilde{J}_0 = \bar{q}_0 \Gamma \tilde{Q}_0$ is the unrenormalized HQET current, $\tilde{Q}_0$ is a two–component static–quark field, satisfying $\gamma_0 \tilde{Q}_0 = \tilde{Q}_0$, and $\tilde{O}_i(\mu)$ are dimension–4 HQET operators, with appropriate quantum numbers. The meaning of the operator equality (1.1) is that on–shell matrix elements of $J(\mu)$, in situations amenable to HQET treatment, after expansion to a given order in $1/m$, coincide with on–shell matrix elements of the right–hand side.

A single dimension–3 term appears on the right–hand side of (1.1) if the Dirac matrix Γ satisfies the conditions

$$\gamma_0 \Gamma = \sigma \Gamma \gamma_0 \quad (\sigma = \pm 1), \quad \gamma_\mu \Gamma \gamma^\mu = 2\sigma h \Gamma, \tag{1.2}$$

where h is a function of the space–time dimension, $d = 4 - 2\varepsilon$. For an antisymmetrized product of n γ–matrices, $\Gamma = \gamma^{[\mu_1} \ldots \gamma^{\mu_n]}$, one obtains $h = \eta(n-2+\varepsilon)$, with $\eta = -\sigma(-1)^n$.

It is natural to perform the matching at a scale $\mu \sim m$, where the matching coefficient $C_\Gamma(\mu)$ contains no large logarithm. One can then use the renormalization group to relate QCD and HQET currents renormalized at arbitrary scales, μ and $\tilde{\mu}$:

$$\exp\left(-\int\limits_{\alpha_s(m)}^{\alpha_s(\mu)} \frac{\gamma_J(\alpha)}{\beta(\alpha)} \frac{d\alpha}{\alpha}\right) J(\mu) = C_\Gamma(m) \exp\left(-\int\limits_{\alpha_s(m)}^{\alpha_s(\tilde{\mu})} \frac{\tilde{\gamma}_J(\alpha)}{\beta(\alpha)} \frac{d\alpha}{\alpha}\right) \tilde{J}(\tilde{\mu}) + \mathrm{O}\left(\frac{1}{m}\right), \tag{1.3}$$

where $\gamma_J = d\log Z_J / d\log\mu$ and $\tilde{\gamma}_J = d\log \tilde{Z}_J / d\log\mu$ are the QCD and HQET current anomalous dimensions, and $\beta = -d\log\alpha_s / d\log\mu$.

To calculate the matching coefficient $C_\Gamma(\mu)$, we consider an on–shell QCD matrix element, $M(\mu) = \left(Z_Q^{\text{os}} Z_q^{\text{os}}\right)^{1/2} Z_J^{-1}(\mu) \Gamma_0$, obtained from the bare proper–vertex function, Γ_0, by renormalizing the current and performing on–shell wave–function renormalization. To zeroth order in $1/m$, it is equal to $C_\Gamma(\mu) \tilde{M}(\mu)$, where $\tilde{M}(\mu) = \left(\tilde{Z}_Q^{\text{os}} Z_q^{\text{os}}\right)^{1/2} \tilde{Z}_J^{-1}(\mu) \tilde{\Gamma}_0$. Hence we obtain

$$C_\Gamma(\mu) = \left(\frac{Z_Q^{\text{os}}}{\tilde{Z}_Q^{\text{os}}}\right)^{1/2} \frac{\tilde{Z}_J(\mu)}{Z_J(\mu)} \frac{\Gamma_0}{\tilde{\Gamma}_0}, \tag{1.4}$$

with a μ dependence coming only from $Z_J(\mu)$ and $\tilde{Z}_J(\mu)$, in agreement with (1.3).

The matching coefficient $C_\Gamma(\mu)$, for any Γ of the form (1.2), was obtained at the one–loop level by Eichten and Hill [1]. The $1/m$ suppressed matching coefficients $B_i(\mu)$ in (1.1) were obtained for vector and axial currents, at one loop, in [4, 5]. Here we shall find the two–loop correction to the leading matching coefficient $C_\Gamma(\mu)$. It is required for a more accurate extraction of QCD matrix elements, such as f_{B} and f_{B^*}, from HQET results obtained, for example, from lattice simulations or sum rules.

As can be seen from (1.3), it is in general necessary to use three–loop anomalous dimensions, γ_J and $\tilde{\gamma}_J$, in conjunction with a two–loop matching coefficient, $C_\Gamma(m)$. Whilst some QCD anomalous dimensions are known at the three–loop level, the universal HQET current anomalous dimension, $\tilde{\gamma}_J$, is currently known only at the one– [6, 7] and two–loop [8, 9, 10] levels. There are, however, three cogent reasons for calculating $C_\Gamma(m)$ to two loops. First, previous experience [11, 12] strongly suggests that two–loop finite effects dominate, numerically, over three–loop anomalous dimensions, near a heavy–quark mass–shell. Secondly, the HQET anomalous dimension is independent of the Dirac structure Γ and hence is absent from important ratios of physical quantities, such as $f_{\mathrm{B}^*}/f_{\mathrm{B}}$. Finally, the three–loop term in $\tilde{\gamma}_J$ may be calculated in future.

Motivated by these considerations, we here compute $C_\Gamma(m)$, to two loops, for an arbitrary Dirac structure of the form (1.2). Section 2 gives our method and general result, for any Γ. In Section 3 we consider ratios of matching coefficients. After elucidating the dependence on γ_5 prescriptions, we show that the two–loop correction to $f_{\mathrm{B}^*}/f_{\mathrm{B}}$, at zeroth order in $1/m$, is comparable in size to each of the effects previously computed, namely the one–loop correction of [1] and the $\mathrm{O}(1/m)$ correction of [13, 14]. In Section 4, we calculate anomalous dimensions and matching coefficients to all orders in α_{s}, in the limit of a large number of flavours, N_f. We extend recent analyses [15, 16, 17] by obtaining the $\overline{\mathrm{MS}}$ matching coefficient for an arbitrary Γ in this limit, confirming the cancellation of renormalon ambiguities in physical matrix elements [17] and validating an additional consistency condition. Section 5 gives a summary of our main results.

2 Two–loop matching calculation

The bare proper vertices, Γ_0 and $\tilde{\Gamma}_0$ in (1.4), may be evaluated at any on–shell momenta of the light and heavy quarks. The calculation is greatly simplified by choosing the momentum of each quark to vanish in HQET. This corresponds to a heavy–quark momentum $p = mv$ in QCD, where $v = (1, \vec{0})$ in the heavy–quark rest frame. One– and two–loop diagrams for the vertex are presented in Fig. 1.

In the quark loop of diagram 1b we include $N_f = N_l + 1$ quark flavours, one of which is the external heavy flavour, whilst the remaining N_l light flavours of quark have masses $m_i < m$, any of which may be zero. In the case of an external b quark, $N_f = 4 + 1 = 5$ and only the mass ratio $m_c/m_b \sim 0.3$ is significantly greater than zero.

We stress that the loop containing the external heavy–quark flavour, with mass $m_i = m$, must be included in all 6 terms on the right–hand side of (1.4). This loop is certainly present in the QCD terms and we must therefore include it the HQET terms, as well. The HQET modification of QCD propagators occurs only along a single heavy–quark line, going through all diagrams; HQET is blind to the contents of loops. In effect, HQET has

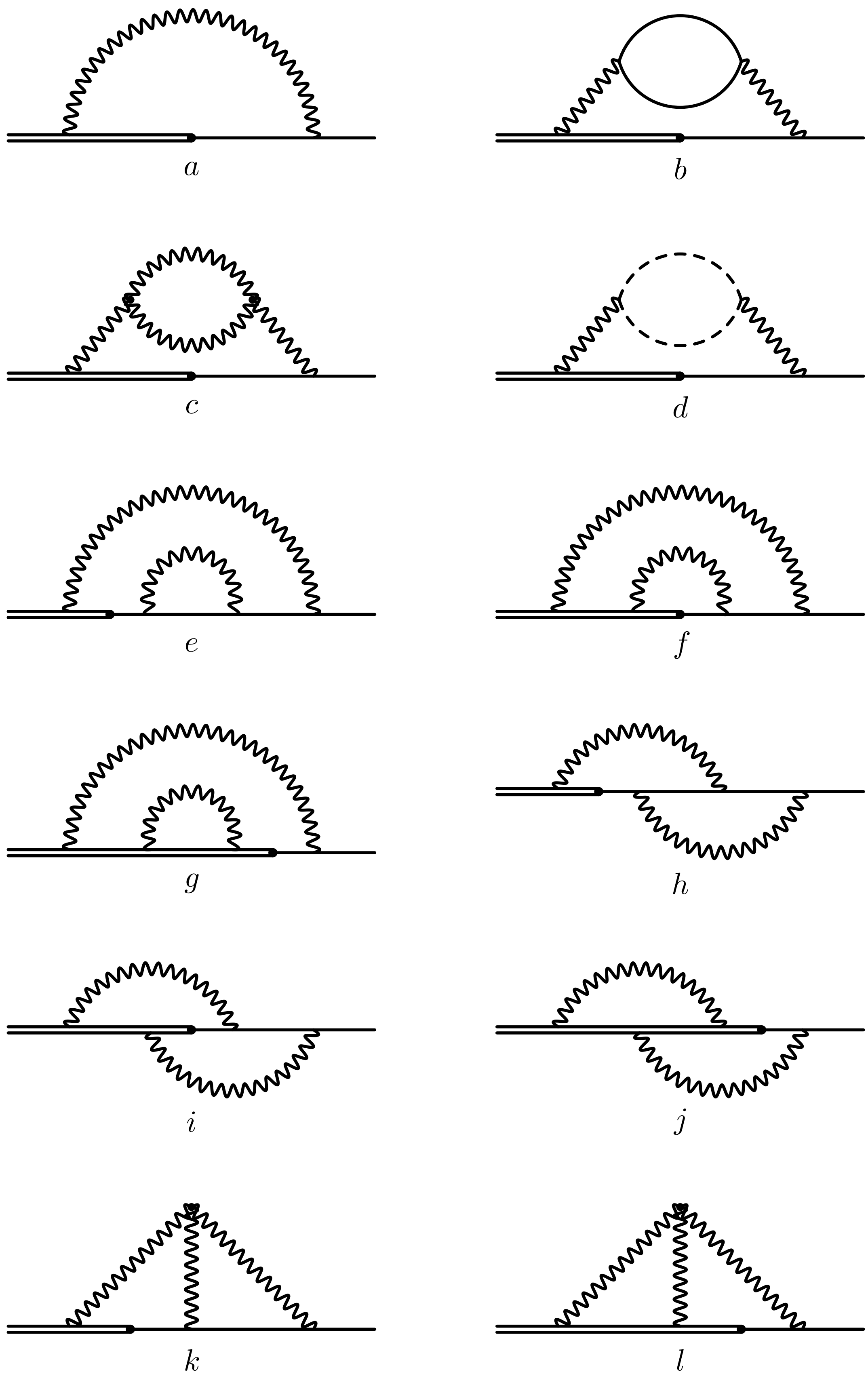

Figure 1: One– and two–loop diagrams for the proper vertex

no memory that the static–quark line had one of the massive flavours that still occur in loop corrections. Just as in conventional QCD, heavy loops decouple from *physical* quantities, at low momenta; not from renormalization constants, or bare vertices. We shall demonstrate the physical decoupling of the t–quark loop from $f_{\mathrm{B}^*}/f_{\mathrm{B}}$ and show that the contribution of the b–quark loop is considerably less than that of a light–quark loop.

We proceed to analyze, in turn, the 3 HQET quantities and 3 QCD quantities on the right–hand side of (1.4).

2.1 HQET calculation

Here we deal with the HQET quantities $\tilde{\Gamma}_0$, $\tilde{Z}_J$, and $\tilde{Z}_Q^{\mathrm{os}}$.

In the case of the HQET proper vertex, $\tilde{\Gamma}_0$, all the dimensionally regularized diagrams of Fig. 1 vanish, for external quarks with zero residual momenta. This is immediately obvious for all except diagram 1b, with a massive–quark loop, since no other diagram contains a scale. In fact, this diagram vanishes too, because the gluon propagator contains a quark–loop insertion and is hence transverse, i. e. proportional to $(g_{\mu\nu} - k_\mu k_\nu/k^2)$. This tensor is contracted with v^ν, from the heavy–quark vertex, and with $\not{k}\gamma^\mu/k^2$, from the light–quark vertex, γ^μ, adjacent to an internal light–quark propagator, $\not{k}/k^2$, with the same momentum as the gluon. The resulting contraction gives $-\vec{\gamma}\cdot\vec{k}\gamma_0/k^2$, which is odd under reflection of the spatial momentum $\vec{k}$ and hence vanishes when integrated over the loop momentum k. Thus $\tilde{\Gamma}_0 = \Gamma_{\mathrm{B}}$, where $\Gamma_{\mathrm{B}} = \bar{u}_q \Gamma u_Q$ is merely the Born term.

The HQET current renormalization $\tilde{Z}_J$ does not depend on Γ. In the $\overline{\mathrm{MS}}$ scheme, at two loops, it is fully determined by the anomalous dimension [8, 9, 10]:

$$\tilde{\gamma}_J = -3C_F\frac{\alpha_{\mathrm{s}}}{4\pi} + C_F\left(\frac{\alpha_{\mathrm{s}}}{4\pi}\right)^2\left[C_F\left(\frac{5}{2} - 16\zeta(2)\right) + C_A\left(-\frac{49}{6} + 4\zeta(2)\right) + \frac{10}{3}T_F N_f\right], \quad (2.1)$$

where $C_A = 3$, $C_F = 4/3$, $T_F = 1/2$, in the case of QCD, and we omit α_{s}^3 terms, here and subsequently, without further comment.

The HQET on–shell wave–function renormalization constant is defined by $\tilde{Z}_Q^{\mathrm{os}} = \left(1 - (\mathrm{d}\tilde{\Sigma}_0/\mathrm{d}\omega)_{\omega=0}\right)^{-1}$ where $\tilde{\Sigma}_0(\omega)$ is the bare static–quark self–energy, at residual energy ω. As in the case of the vertex, only diagrams containing a loop with a quark of non–zero mass can contribute, since no other diagram contains a scale. Thus we obtain

$$\left.\frac{\mathrm{d}\tilde{\Sigma}_0}{\mathrm{d}\omega}\right|_{\omega=0} = -iC_F g_0^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{v^\mu v^\nu}{(vk)^2}\frac{\Pi_{\mu\nu}(k)}{k^4} = -i\frac{C_F g_0^2}{d-1}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\left(\frac{k^2}{(vk)^2} - 1\right)\frac{\Pi_\alpha^\alpha(k)}{k^6}, \quad (2.2)$$

where $\Pi_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu/k^2)\Pi_\alpha^\alpha(k)/(d-1)$ gives the contribution of quark loops to the polarization operator, whose trace depends only on k^2. We now average over all directions of k, in d–dimensional space, obtaining $< k^2/(vk)^2 >= 2-d$. A dimensionally regularized result is then readily obtained from two–loop massive bubble integrals [11], which give

$$\left.\frac{\mathrm{d}\tilde{\Sigma}_0}{\mathrm{d}\omega}\right|_{\omega=0} = -8C_F T_F\frac{g_0^4}{(4\pi)^d}\sum_{m_i>0} m_i^{-4\varepsilon}\frac{(d-1)(d-6)\Gamma^2(1+\varepsilon)}{(d-2)(d-4)^2(d-5)(d-7)}, \quad (2.3)$$

whose Laurent expansion yields the on–shell wave–function renormalization constant

$$\tilde{Z}_Q^{\rm os} = 1 + 2C_F T_F \left(\frac{\alpha_{\rm s}}{4\pi}\right)^2 \sum_{m_i>0} \left(\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}\log\frac{m_i}{\mu} - \frac{4}{3\varepsilon} + 8\log^2\frac{m_i}{\mu} + \frac{16}{3}\log\frac{m_i}{\mu} + \zeta(2) + \frac{26}{9}\right). \tag{2.4}$$

We note that (2.4) exhibits non–uniformity as $m_i \to 0$: quarks with masses $m_i > 0$ are to be included; any quark with $m_i = 0$ is excluded, by dimensional regularization. This non–uniformity has the same infrared origin as that observed in on–shell QCD wave–function renormalization [12], to which we now turn.

2.2 QCD calculation

For continuity, we deal with the QCD quantities, Γ_0, Z_J, and $Z_Q^{\rm os}$, in reverse order.

A gauge–invariant on–shell wave–function renormalization constant, $Z_Q^{\rm os}$, was obtained in d dimensions, at the two–loop level, in [12]. A quark flavour with a small but non–zero mass gives a contribution to $Z_Q^{\rm os}$ different from that of a zero–mass flavour. The same is seen to be true for $\tilde{Z}_Q^{\rm os}$ in (2.4). Each discontinuity originates from the infrared region of small gluon momenta, where HQET does not differ from QCD. Therefore these discontinuities should cancel in the ratio $Z_Q^{\rm os}/\tilde{Z}_Q^{\rm os}$, which we indeed find to have a smooth limit, as $m_i \to 0$. This uniformity provides a strong check of the massless and massive quark–loop contributions to $Z_Q^{\rm os}$, obtained in [12]. The ratio

$$\frac{Z_Q^{\rm os}/\tilde{Z}_Q^{\rm os}}{(Z_Q^{\rm os}/\tilde{Z}_Q^{\rm os})_{m_i=0}} = 1 + 2C_F T_F \left(\frac{\alpha_{\rm s}}{4\pi}\right)^2 \sum_{i=1}^{N_f} \Delta_Z\left(\frac{m_i}{m}\right) \tag{2.5}$$

may be expressed as an integral of the difference, $\Pi\left(-m_i^2/k^2\right)$, between the polarization operator with a quark of mass m_i in the loop and that with a massless quark. This difference contains neither ultraviolet nor infrared divergences. The integral over the gluon momentum k is likewise well–behaved, and may be performed in 4 dimensions. As is the case for all two–scale calculations in this paper, the result may be expressed as a combination of three basic dilogarithmic integrals, defined and evaluated in Appendix A. Combining $Z_Q^{\rm os}$, from [12], with $\tilde{Z}_Q^{\rm os}$, from (2.4), we obtain

$$\Delta_Z(r) = -4\Delta_1(r) - 2\Delta_2(r) - 12\Delta_3(r)\,. \tag{2.6}$$

The QCD current renormalization constants Z_J are known to two loops for some, but not all, of the currents that we study. We shall simply apply a minimal Ansatz for Z_J and extract its coefficients from the requirement of finiteness of (1.4), checking our resultant master formula for γ_J against known special cases.

The most difficult part of the problem is the calculation of the bare QCD proper vertex, Γ_0. We shall first find it in the case when all light flavours are massless, and include later the effects of finite light–quark masses. Since we calculate the bare vertex on the renormalized mass–shell, $p = mv$, it is convenient to express the result in terms of the on–shell mass m, treating the one–loop d–dimensional counterterm, $\Delta m = m - m_0$, as a perturbation.

Initially, we make no assumption about the properties of the matrix Γ, and note that each two–loop diagram for Γ_0 may be written as a sum of terms, each of the form

$$\bar{u}_q\gamma_{\mu_1}\ldots\gamma_{\mu_l}\Gamma\gamma_{\nu_1}\ldots\gamma_{\nu_r}u_Q\cdot I^{\mu_1\ldots\mu_l;\nu_1\ldots\nu_r}\,, \tag{2.7}$$

where I is some integral over loop momenta, l is even, and $l+r\leq 8$. After the integration, $I^{\mu_1\ldots\mu_l;\nu_1\ldots\nu_r}$ can contain only $g^{\mu\nu}$ and v^α. Resulting contractions of pairs of γ–matrices on the left, and of pairs on the right, merely produce additional terms of the form (2.7), with smaller values of $l+r$. Before performing the remaining contractions, one may anticommute γ–matrices, so as to arrange that $\not{v}$ occurs only on the extreme left, or the extreme right, with the contracted indices in between occurring in opposite orders on the left and right of Γ. Additional terms, arising from anticommutators, have fewer γ–matrices, with l remaining even. Repeating this procedure for all values of $l+r$, from 8 down to 0, we may cast any diagram in the form

$$\begin{aligned}D=\bar{u}_q[&\Gamma(x_0+x_1\not{v})+\not{v}\gamma_\alpha\Gamma\gamma^\alpha(x_2+x_3\not{v})+\gamma_\alpha\gamma_\beta\Gamma\gamma^\beta\gamma^\alpha(x_4+x_5\not{v})\\&+\not{v}\gamma_\alpha\gamma_\beta\gamma_\gamma\Gamma\gamma^\gamma\gamma^\beta\gamma^\alpha(x_6+x_7\not{v})+\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\Gamma\gamma^\delta\gamma^\gamma\gamma^\beta\gamma^\alpha x_8]u_Q\,.\end{aligned} \tag{2.8}$$

Now we assume that the matrix Γ has the properties (1.2). The effect of each contraction is then to produce a factor $2\sigma h$. Terms with an odd number of contractions necessarily contain $\not{v}$ on the left, which yields an extra σ when moved to the right, where it merely gives $\not{v}u_Q=u_Q$. Thus we obtain a result involving only powers of h:

$$D=\Big[(x_0+x_1)+(x_2+x_3)(2h)+(x_4+x_5)(2h)^2+(x_6+x_7)(2h)^3+x_8(2h)^4\Big]\,\Gamma_{\rm B}\,. \tag{2.9}$$

We can find the coefficients x_0+x_1, x_2+x_3, x_4+x_5, x_6+x_7, x_8, for each diagram, by taking, separately, a trace of the γ–matrices on the light–quark line with L_i, and a trace on the heavy–quark line with H_i, using the following 5 forms of $L_i\times H_i$: $1\times(1+\not{v})$; $\gamma_\mu\not{v}\times(1+\not{v})\gamma^\mu$; $\gamma_\mu\gamma_\nu\times(1+\not{v})\gamma^\nu\gamma^\mu$; $\gamma_\mu\gamma_\nu\gamma_\rho\not{v}\times(1+\not{v})\gamma^\rho\gamma^\nu\gamma^\mu$; $\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\times(1+\not{v})\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu$. Performing the same operation on the generic form (2.8), we obtain 5 equations relating these double–traces to the desired coefficients. Inverting these equations, once and for all, we may then convert the *integrand* of any diagram into a 4th order polynomial in h, with coefficients that are scalar functions of v and the two loop momenta. For reliability, we checked our general result, for each two–loop diagram, against brute–force evaluation of 8 specific cases of Dirac matrix. The general one–loop diagram 1a, and its associated mass–counterterm contribution, were similarly reduced to quadratic functions of h.

All the resulting one–loop scalar integrals can be expressed in terms of

$$I_0=-\frac{i}{\pi^{d/2}}\int\frac{{\rm d}^dk}{k^2+2vk}=\frac{\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)}\,. \tag{2.10}$$

The two–loop integrals may be reduced to combinations of I_0^2 and two further terms:

$$\begin{aligned}I_1&=-\frac{1}{\pi^d}\int\frac{{\rm d}^dk{\rm d}^dl}{k^2(l-k)^2(l^2+2vl)}\\&=\frac{1-4\varepsilon}{2\varepsilon^2(1-2\varepsilon)(1-\frac{3}{2}\varepsilon)(1-3\varepsilon)}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma(1-4\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1-3\varepsilon)}\,,\\I_2&=-\frac{1}{\pi^d}\int\frac{{\rm d}^dk{\rm d}^dl}{(k^2+2vk)(l^2+2vl)((k+l)^2+2v(k+l))}\\&=\frac{3(5d-18)(d-2)^2}{2(3d-8)(3d-10)(d-3)}I_0^2-\frac{2(d-4)}{2d-7}I_1-\frac{16(d-4)^2}{(3d-8)(3d-10)}I(\varepsilon)\,,\end{aligned} \tag{2.11}$$

where [18, 19]

$$I(\varepsilon) = I + \mathrm{O}(\varepsilon), \quad I = \pi^2 \log 2 - \tfrac{3}{2}\zeta(3)\,. \tag{2.12}$$

To achieve this reduction, we used the package RECURSOR [19], written in REDUCE [20] to implement recurrence relations derived from integration by parts [11, 12].

So as to have a strong check, we performed the calculation in an arbitrary covariant gauge, verifying that the sum of all diagrams for the on–shell vertex, including the mass counterterm, is gauge invariant in d dimensions. The result can be written as

$$\frac{\Gamma_0}{\Gamma_{\rm B}} = 1 - C_F \frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}} I_0 \frac{(1-h)(d-2+2h)}{2(d-3)} + C_F \frac{g_0^4 m^{-4\varepsilon}}{(4\pi)^d} \sum_{i=0}^{3}\sum_{j=0}^{2}\sum_{k=0}^{3} a_{jik} C_i J_j h_k\,, \tag{2.13}$$

in terms of 4 colour factors, 3 integral structures, and 4 current–specific functions. The colour factors are chosen as follows: $C_0 = C_F - \frac{1}{2}C_A$; $C_1 = C_F$; $C_2 = T_F N_l$, from N_l massless loops in diagram 1b; $C_3 = T_F$, from the heavy–quark loop. The integrals

$$\begin{aligned} J_0 &= \frac{(d-2)I_0^2}{8(d-1)(d-3)^2(d-4)^2(d-5)(d-6)}\,, \\ J_1 &= \frac{I_1}{8(d-1)(d-3)(d-4)(2d-7)}\,, \quad J_2 = \frac{I_2}{4(d-1)(d-4)^2(d-6)}\,, \end{aligned} \tag{2.14}$$

are chosen to make the coefficients a_{jik} polynomials in d. All dependence on Γ resides in $h_k = h^k$, for $k = 0, 1, 2$, and $h_3 = h^3(h + d - 4)$. Only diagrams 1f and 1i have sufficient γ–matrices on each side of the vertex to generate h^3 and h^4 terms, and we find that each diagram produces the combination h_3, for each integral structure. Moreover, the integral J_2 arises only from diagram 1j, with colour factor C_0, or from the heavy–quark loop in diagram 1b, with colour factor C_3. Thus several of the 48 coefficients a_{jik} vanish. The 28 that survive are listed in Appendix B.

We complete the calculation by including the effect of non–zero light–quark masses. The difference of diagram 1b, with a quark of mass m_i in the loop, and the same diagram with a massless quark, contains neither ultraviolet nor infrared divergences. Therefore we calculate it in 4 dimensions, obtaining a further term, to be added to (2.13), of the form

$$\frac{\Delta\Gamma_0}{\Gamma_{\rm B}} = C_F T_F \left(\frac{\alpha_{\rm s}}{4\pi}\right)^2 \sum_{i=1}^{N_l} \Delta_\Gamma\left(\frac{m_i}{m}\right)\,, \tag{2.15}$$

where the sum runs over light flavours only, and Δ_Γ can be reduced to the integrals (A.2), as follows:

$$\Delta_\Gamma(r) = \frac{4}{3}\left[-2(1-h^2)\Delta_1(r) - (1+2h-2h^2)\Delta_2(r) + 2h(2-h)\Delta_3(r)\right]\,, \tag{2.16}$$

with $\Delta_\Gamma(0) = 0$, by virtue of its definition. As a strong check on the coefficients of the colour factors C_3 and C_2 in (2.13), coming from loops with quarks of masses $m_i = m$ and $m_i = 0$, respectively, we have verified that setting $d = 4$ in their difference agrees with setting $r = 1$ in (2.16).

2.3 Results

We re–express the bare QCD vertex (2.13) in terms of the $\overline{\rm MS}$ coupling, $\alpha_{\rm s}(\mu)$, using one–loop coupling–constant renormalization, with $N_f = N_l + 1$ flavours, and expand the result as a Laurent series in ε. Using $Z_Q^{\rm os}$ from [12], $\tilde\Gamma_0 = \Gamma_{\rm B}$, $\tilde Z_J(\mu)$ from (2.1), and $\tilde Z_Q^{\rm os}$ from (2.4), we obtain the generic QCD current renormalization constant, $Z_J(\mu)$, by requiring the finiteness of (1.4). The corresponding anomalous dimension is that of the QCD current $J(\mu) = \bar q \gamma^{[\mu_1} \ldots \gamma^{\mu_n]} q$, which does not depend upon whether the quarks are heavy or light. We find that

$$\begin{aligned}\gamma_J = {}& -C_F \frac{\alpha_{\rm s}}{2\pi}(n-1)(n-3)\left[1 + \frac{\alpha_{\rm s}}{4\pi}\left(C_F \frac{5(n-2)^2 - 19}{2} - C_A \frac{3(n-2)^2 - 19}{3}\right)\right] \\ & - C_F \left(\frac{\alpha_{\rm s}}{4\pi}\right)^2 (n-1)(n-15)\frac{11 C_A - 4 T_F N_f}{9}\,, \qquad (2.17)\end{aligned}$$

with a check being provided by the absence of the signature η, which merely distinguishes different components of J. We reproduce the known result for the scalar current [21], with $n = 0$, and, of course, obtain a vanishing result for the vector current, with $n = 1$. The two–loop anomalous dimension of the $n = 2$ current, $\bar q \sigma_{\mu\nu} q$, appears to be absent from the literature, though it could be derived by straightforward calculation in massless QCD. Instead, we have obtained

$$\gamma_J\big|_{n=2} = C_F \frac{\alpha_{\rm s}}{2\pi}\left[1 + \frac{\alpha_{\rm s}}{4\pi}\left(-\frac{19}{2} C_F + \frac{257}{18} C_A - \frac{26}{9} T_F N_f\right)\right] \qquad (2.18)$$

as a by–product of a more difficult massive on–shell calculation. The currents with $n = 3$ and $n = 4$ are non–singlet axial and pseudoscalar currents, with the 't Hooft–Veltman γ_5, whose anomalous dimensions are known to differ, at two loops [22, 23] and beyond [23, 24], from those with the naively anticommuting γ_5, with $n = 1$ and $n = 0$, respectively. This difference is apparent in the final term of (2.17), which lacks invariance under $n \to 4 - n$. Our results for $n = 3$ and $n = 4$ agree with [22] and [23], respectively.

Finally we arrive at the general expression for the two–loop matching coefficient

$$\begin{aligned}C_\Gamma(m) = {}& 1 + C_F \frac{\alpha_{\rm s}(m)}{4\pi}\left[3(n-2)^2 + (2-\eta)(n-2) - 4\right] \\ & + C_F \left(\frac{\alpha_{\rm s}}{4\pi}\right)^2 \left[C_F a_F + C_A a_A + T_F \sum_{i=1}^{N_f}\left\{a_f + \Delta_J\left(\frac{m_i}{m}\right)\right\}\right]\,, \qquad (2.19)\end{aligned}$$

where the sum now *includes* the heavy flavour, with $m_i = m$. The one–loop term coincides with [1]. The coefficients in the two–loop term are

$$\begin{aligned}a_F = {}& \left(\frac{317}{24} - \frac{10}{3}\zeta(2)\right)(n-2)^4 + 11(n-2)^3 - \frac{11}{2}\eta(n-2)^3 \\ & + \left(-\frac{253}{6} + 48\zeta(2) - \frac{16}{3} I\right)(n-2)^2 - 2\eta(n-2)^2 - 20(n-2) \\ & + \left(\frac{32}{3} - \frac{64}{3}\zeta(2) + \frac{8}{3} I\right)\eta(n-2) + \frac{689}{16} - 81\zeta(2) - 8\zeta(3) + 12 I\,, \\ a_A = {}& \left(-\frac{43}{12} + \frac{4}{3}\zeta(2)\right)(n-2)^4 - 2(n-2)^3 + \eta(n-2)^3 \qquad (2.20)\end{aligned}$$

$$+\left(\frac{9491}{216}-\frac{52}{3}\zeta(2)+\frac{8}{3}I\right)(n-2)^2+\frac{143}{18}(n-2)$$
$$+\left(-\frac{281}{18}+8\zeta(2)-\frac{4}{3}I\right)\eta(n-2)-\frac{29017}{432}+29\zeta(2)+2\zeta(3)-6I\,,$$
$$a_f=\left(-\frac{445}{54}-\frac{8}{3}\zeta(2)\right)(n-2)^2-\frac{2}{9}(n-2)+\frac{38}{9}\eta(n-2)+\frac{1745}{108}+\frac{20}{3}\zeta(2)\,.$$

The mass correction $\Delta_J(r)=\Delta_\Gamma(r)+\Delta_Z(r)$ depends only on $h|_{\varepsilon=0}=\eta(n-2)$, because (2.6) and (2.16) result from finite, 4–dimensional integrals. Its general form is

$$\Delta_J(r)=\frac{2}{3}\Big[2\left(2(n-2)^2-5\right)\Delta_1(r)+\left(4(n-2)^2-4\eta(n-2)-5\right)\Delta_2(r)$$
$$-2\left(2(n-2)^2-4\eta(n-2)+9\right)\Delta_3(r)\Big]\,. \qquad (2.21)$$

The heavy–flavour contribution, with $m_i=m$, is proportional to $a_f+\Delta_J(1)$, where

$$\Delta_J(1)=\frac{2}{3}\left[2(7+2\zeta(2))(n-2)^2+4(-5+2\zeta(2))\eta(n-2)+17-38\zeta(2)\right] \qquad (2.22)$$

is always opposite in sign to a_f and close to it in magnitude, resulting in a heavy–flavour contribution that is always small, in comparison with that from a light flavour.

Table 1: Matching coefficients

Γ	$C_\Gamma(m)$
1	$1+\frac{2}{3}\frac{\alpha_s(m)}{\pi}+(10.92-0.60N_l+0.04)\left(\frac{\alpha_s}{\pi}\right)^2$
γ_0	$1-\frac{2}{3}\frac{\alpha_s(m)}{\pi}-(4.20-0.44N_l+0.07)\left(\frac{\alpha_s}{\pi}\right)^2$
γ_1	$1-\frac{4}{3}\frac{\alpha_s(m)}{\pi}-(11.50-0.79N_l+0.09)\left(\frac{\alpha_s}{\pi}\right)^2$
$\gamma_0\gamma_1$, $\gamma_1\gamma_2$	$1-\frac{4}{3}\frac{\alpha_s(m)}{\pi}-(16.19-1.13N_l+0.13)\left(\frac{\alpha_s}{\pi}\right)^2$
$\gamma_0\gamma_1\gamma_2$	$1-(10.98-0.77N_l+0.11)\left(\frac{\alpha_s}{\pi}\right)^2$
$\gamma_1\gamma_2\gamma_3$	$1+\frac{2}{3}\frac{\alpha_s(m)}{\pi}-(2.78-0.42N_l+0.08)\left(\frac{\alpha_s}{\pi}\right)^2$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$1+\frac{10}{3}\frac{\alpha_s(m)}{\pi}+(19.75-0.64N_l+0.00)\left(\frac{\alpha_s}{\pi}\right)^2$

There are 8 distinct matrices, Γ, in 4–dimensional space–time: antisymmetrized products of up to 3 spatial γ–matrices, and the same products multiplied by γ_0. For each current, we obtain a matching coefficient $C_\Gamma(m)$, by choosing appropriate values of n and η in the formulæ (2.19) to (2.22). With N_l zero–mass quarks, and QCD colour factors, we obtain the numerical values of Table 1, two of which coincide (for reasons explained in the next section). Typical matrices Γ are presented in the first column; $C_\Gamma(m)$ is, of course, the same for other spatial components. Coefficients of $(\alpha_s/\pi)^2$ in the two–loop corrections are written as sums of three terms: the quenched contribution, without quark loops; the light–quark loop contribution; and the heavy–quark loop contribution. One can see that the quenched contribution is usually of order 10; each light flavour gives a contribution of order 1, with the opposite sign; and the external flavour contributes of order 0.1. In most cases, the two–loop correction has the same sign as the one–loop correction.

3 Ratios of matching coefficients

First we consider ratios that reflect differences in γ_5 prescriptions. Then we analyze ratios of meson matrix elements.

3.1 't Hooft–Veltman versus anticommuting γ_5

It is generally accepted that there exists no d–dimensional generalization of γ_5 that anticommutes with all γ_μ. Nor is there any covariant d–dimensional generalization of the Levi–Civita tensor, $\varepsilon_{\mu\nu\sigma\rho}$. Instead, one may use the 't Hooft–Veltman γ_5,

$$\gamma_5^{\mathrm{HV}} = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{3.1}$$

which makes multi–loop calculation rather time–consuming, even when implemented covariantly, using antisymmetrized products [23, 24, 25, 26]. However, there is also a general belief that one may use a naively anticommuting matrix, γ_5^{AC}, in open fermion lines, and in loops with an even number of γ_5–matrices, without encountering contradictions.

The antisymmetrized products of n and $4-n$ γ–matrices are related to each other by multiplication by γ_5^{HV}, and have different matching coefficients. On the other hand, multiplying Γ by the naively anticommuting γ_5^{AC} does not change h in (1.2), and hence cannot change the matching coefficients. Therefore ratios of matching coefficients with n and $4-n$ antisymmetrized γ–matrices express differences between using the different γ_5 prescriptions.

HQET matrix elements do not depend on the γ_5 prescription. For example,

$$<0|\bar{q}\gamma_5^{\mathrm{HV}}\tilde{Q}|\mathrm{B}> = <0|\bar{q}\gamma_5^{\mathrm{AC}}\tilde{Q}|\mathrm{B}>\,. \tag{3.2}$$

To prove this equality, consider the HQET operator–product expansion [27] for the correlator of two pseudoscalar currents. In every contributing diagram, the static–quark propagator between the two γ_5–vertices has the γ–matrix structure $(1+\gamma_0)/2$, which is unaffected by gluon vertices. Since every sort of γ_5–matrix anticommutes with γ_0, we can move one γ_5–matrix from its vertex and annihilate it with the other, leaving $(1-\gamma_0)/2$ on the heavy–quark line, independently of the prescription. Since the correlator is independent of the prescription, so is its spectral density, and in particular the ground–state contribution. Hence we arrive at (3.2).

In contrast to this, the QCD matrix elements do *not* coincide. We have already remarked that the $\overline{\mathrm{MS}}$–renormalized pseudoscalar currents, $J^{\mathrm{HV}}(\mu) = Z_{\mathrm{HV}}^{-1}(\mu)\bar{q}_0\gamma_5^{\mathrm{HV}}Q_0$ and $J^{\mathrm{AC}}(\mu) = Z_{\mathrm{AC}}^{-1}(\mu)\bar{q}_0\gamma_5^{\mathrm{AC}}Q_0$, have anomalous dimensions that differ, at two loops and beyond. The currents are related to each other by a finite renormalization, $J^{\mathrm{AC}}(\mu) = Z_P(\mu)J^{\mathrm{HV}}(\mu)$, whose term of order α_s^L may be found either by comparing renormalized matrix elements, at L loops, or more demandingly, by equating $\mathrm{d}\log Z_P/\mathrm{d}\log\mu$ to the difference, $\gamma_{J^{\mathrm{HV}}} - \gamma_{J^{\mathrm{AC}}}$, of anomalous dimensions, at $L+1$ loops. The consistency of these methods has been demonstrated, at $L=2$, by evaluation of massless three–loop diagrams [23], which yield

$$Z_P(\mu) = 1 - 2C_F\frac{\alpha_s(\mu)}{\pi} + C_F\left(\frac{\alpha_s}{4\pi}\right)^2\frac{2(C_A+4T_FN_f)}{9} + \mathrm{O}(\alpha_s^3)\,, \tag{3.3}$$

using either method, with a further $O(\alpha_s^3)$ term determined solely by the first method. Consistency demands that this result be obtainable by comparing matrix elements with quarks of any mass and momenta, and in particular that

$$Z_P(\mu)=\frac{<0|J^{\mathrm{AC}}(\mu)|\mathrm{B}>}{<0|J^{\mathrm{HV}}(\mu)|\mathrm{B}>}=\frac{C_1(\mu)}{C_{\gamma_0\gamma_1\gamma_2\gamma_3}(\mu)}\,,\tag{3.4}$$

given the equality (3.2). Setting $\mu=m$ and using (2.19), we indeed verify (3.3). In particular, the two–loop finite–mass corrections (2.21) cancel in the ratio (3.4), as should occur in a result obtainable [23] from three–loop mass–independent anomalous dimensions.

Similarly, the $\overline{\mathrm{MS}}$–renormalized 't Hooft–Veltman and naively anticommuting non–singlet axial currents are related by a finite renormalization, $J_\mu^{\mathrm{AC}}(\mu)=Z_A(\mu)J_\mu^{\mathrm{HV}}(\mu)$, which has been computed at the one– [22], two– [25], and three–loop [23, 24] levels. We are informed by S. A. Larin that the discrepancy between [23, 24] and [25] at two loops is attributable to the use of the so–called G–scheme in [25]. The $\overline{\mathrm{MS}}$ result is [23, 24]

$$Z_A(\mu)=1-C_F\frac{\alpha_s(\mu)}{\pi}+C_F\left(\frac{\alpha_s}{4\pi}\right)^2\frac{198C_F-107C_A+4T_FN_f}{9}+O(\alpha_s^3)\,,\tag{3.5}$$

which we verified by calculation of *two* ratios

$$Z_A(\mu)=\frac{C_{\gamma_0}(\mu)}{C_{\gamma_1\gamma_2\gamma_3}(\mu)}=\frac{C_{\gamma_3}(\mu)}{C_{\gamma_0\gamma_1\gamma_2}(\mu)}\,.\tag{3.6}$$

Note that the weak axial current is J_μ^{AC}. Only for this current does QCD renormalization preserve a $V-A$ structure. Thus measurable matrix elements, such as f_{B}, are obtained from J_μ^{AC}, not from the chiral–symmetry–breaking 't Hooft–Veltman counterpart.

The QCD tensor current, $J_{\mu\nu}=\bar{q}\sigma_{\mu\nu}q$, is a special case, because inclusion of γ_5^{HV}, to form $J_{\mu\nu}^{\mathrm{HV}}=\bar{q}\gamma_5^{\mathrm{HV}}\sigma_{\mu\nu}q$, is merely a space–time transformation, in 4 dimensions, giving, for example, $J_{01}^{\mathrm{HV}}=-\mathrm{i}J_{23}$. Thus there can be no difference of anomalous dimensions between $J_{\mu\nu}^{\mathrm{HV}}$ and $J_{\mu\nu}^{\mathrm{AC}}$, and hence no non–trivial finite renormalization, $Z_T(\mu)$, relating the two prescriptions. Equivalently, we have *two* methods of obtaining $Z_T(\mu)$:

$$Z_T(\mu)=\frac{C_{\gamma_0\gamma_1}(\mu)}{C_{\gamma_2\gamma_3}(\mu)}=\frac{C_{\gamma_2\gamma_3}(\mu)}{C_{\gamma_0\gamma_1}(\mu)}\,,\tag{3.7}$$

which proves both that $Z_T(\mu)=1$ and that $C_{\gamma_0\gamma_1}(m)=C_{\gamma_2\gamma_3}(m)$, as found from (2.19). It is easy to see how this equality comes about, purely within our calculation of matching coefficients. With $n=2$, we have $h=\eta\varepsilon$, so that the only possible origin of a difference between $C_{\gamma_0\gamma_1}(\mu)$ and $C_{\gamma_2\gamma_3}(\mu)$ would be a term h/ε in Γ_0. Such a term would give an η–dependent QCD anomalous dimension, for $n\neq 2$, and hence cannot occur.

3.2 Ratios of meson matrix elements

After eliminating the currents containing the 't Hooft–Veltman γ_5, we are left with 4 essentially different currents. In this subsection, we discuss the ground–state–meson matrix

elements:

$$\begin{aligned}
&{<}0|(\bar{q}\gamma_5 Q)_\mu|\mathrm{B}{>} = -\mathrm{i}\, m_\mathrm{B} f^P_\mathrm{B}(\mu)\,,\\
&{<}0|\bar{q}\gamma_\alpha\gamma_5 Q|\mathrm{B}{>} = \mathrm{i}\, f_\mathrm{B} p_\alpha\,,\\
&{<}0|\bar{q}\gamma_\alpha Q|\mathrm{B}^*{>} = \mathrm{i}\, m_{\mathrm{B}^*} f_{\mathrm{B}^*} e_\alpha\,,\\
&{<}0|(\bar{q}\sigma_{\alpha\beta} Q)_\mu|\mathrm{B}^*{>} = f^T_{\mathrm{B}^*}(\mu)(p_\alpha e_\beta - p_\beta e_\alpha)\,,
\end{aligned} \tag{3.8}$$

where $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$, and we use, from now on, the naively anticommuting γ_5 that is appropriate to matrix elements of non–singlet weak currents. The corresponding HQET quantities coincide: $\tilde{f}^P_\mathrm{B} = \tilde{f}_\mathrm{B} = \tilde{f}_{\mathrm{B}^*} = \tilde{f}^T_{\mathrm{B}^*}$, as consequences of the heavy–quark symmetry. Hence, ratios of QCD matrix elements are equal to ratios of matching coefficients. These ratios do not depend on the unknown three–loop HQET anomalous dimension. Similar formulæ hold for P–wave 0^+, 1^+ mesons, if one inserts an extra γ_5 into all currents.

It follows from the equations of motion that

$$\frac{f^P_\mathrm{B}(\mu)}{f_\mathrm{B}} = \frac{{<}0|(\bar{q}\gamma_5 Q)_\mu|\mathrm{B}{>}}{{<}0|\bar{q}\gamma_5\gamma_0 Q|\mathrm{B}{>}} = \frac{m_\mathrm{B}}{\overline{m}(\mu)}\,, \tag{3.9}$$

where $\overline{m}(\mu)$ is the $\overline{\mathrm{MS}}$ running heavy–quark mass. To leading order in $1/m$, we may replace m_B by the on–shell mass m, obtaining

$$\begin{aligned}
\frac{f^P_\mathrm{B}(m)}{f_\mathrm{B}} &= \frac{m}{\overline{m}(m)} = \frac{C_1(m)}{C_{\gamma_0}(m)} = 1 + C_F\frac{\alpha_\mathrm{s}(m)}{\pi}\\
&C_F\left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2\left[C_F\left(\frac{121}{8} + 30\zeta(2) - 8I\right) + C_A\left(\frac{1111}{24} - 8\zeta(2) + 4I\right)\right.\\
&\left.+ 8T_F\sum_{i=1}^{N_f}\left\{-\frac{71}{48} - \zeta(2) + \Delta_1\left(\frac{m_i}{m}\right) + \Delta_3\left(\frac{m_i}{m}\right)\right\}\right]\\
&\approx 1 + \frac{4}{3}\frac{\alpha_\mathrm{s}(m)}{\pi} + (16.01 - 1.04N_l + 0.10)\left(\frac{\alpha_\mathrm{s}}{\pi}\right)^2\,,
\end{aligned} \tag{3.10}$$

where the final numerical result shows, as in Table 1, the contributions of: the quenched term; N_l massless–quark loops; and the heavy–quark loop. We confirm the analytical result of [11], in the case of $N_l = N_f - 1$ massless quarks, and correct the omission of a factor $C_F = 4/3$ from the numerically much smaller effects of finite light–quark masses in Equation (17) of [11].

Our most important result is for the ratio of two observable matrix elements:

$$\frac{{<}0|\bar{q}\vec{\gamma}Q|\mathrm{B}^*{>}}{{<}0|\bar{q}\gamma_0\gamma_5 Q|\mathrm{B}{>}} = \frac{m_{\mathrm{B}^*} f_{\mathrm{B}^*}\vec{e}}{m_\mathrm{B} f_\mathrm{B}}\,, \tag{3.11}$$

which is independent of the renormalization scale. At leading order in $1/m$, we obtain

$$\begin{aligned}
\frac{f_{\mathrm{B}^*}}{f_\mathrm{B}} &= \frac{C_{\gamma_1}(m)}{C_{\gamma_0}(m)} = 1 - C_F\frac{\alpha_\mathrm{s}(m)}{2\pi}\\
&+ C_F\left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2\left[C_F\frac{1}{3}(31 - 128\zeta(2) + 16I) + C_A\frac{1}{9}(-263 + 144\zeta(2) - 24I)\right.\\
&\left.- \frac{16}{3}T_F\sum_{i=1}^{N_f}\left\{-\frac{19}{12} + \Delta_2\left(\frac{m_i}{m}\right) - 2\Delta_3\left(\frac{m_i}{m}\right)\right\}\right]\\
&\approx 1 - \frac{2}{3}\frac{\alpha_\mathrm{s}}{\pi} - (7.75 - 0.35N_l + 0.03)\left(\frac{\alpha_\mathrm{s}}{\pi}\right)^2.
\end{aligned} \tag{3.12}$$

Note that this is numerically very close to $(\overline{m}(m)/m)^{1/2}$, as can be seen from (3.10).

At any finite order of perturbation theory, the ratio (3.12) is determined by a combination of loop integrals that contains neither infrared nor ultraviolet divergences. Therefore all loop momenta are of order m. In order to demonstrate the absence of infrared sensitivity (which may sometimes be overlooked in dimensional regularization) we repeated the one–loop calculation with a gluon mass, λ, finding that the only effect is to multiply the $\mathrm{O}(\alpha_\mathrm{s})$ term in (3.12) by

$$G\left(\frac{\lambda}{m}\right) = \frac{1}{3}\int_0^\infty \frac{\mathrm{d}k^2}{k^2+\lambda^2} F\left(\frac{k^2}{m^2}\right) = 1 - \frac{2\pi}{3}\frac{\lambda}{m} + \mathrm{O}\left(\frac{\lambda^2}{m^2}\right), \tag{3.13}$$
$$F(x) = (x+1)\sqrt{x(x+4)} - x(x+3) = \begin{cases} 2\sqrt{x}, & x \ll 1\,, \\ 2/x, & x \gg 1\,. \end{cases}$$

In general, we find that all ratios of matching coefficients are infrared–safe, though those involving QCD currents with anomalous dimensions clearly require ultraviolet regularization.

We now demonstrate that the existence of a very heavy flavour, such as top, has *no* effect on the observable ratio $f_{\mathrm{B}^*}/f_\mathrm{B}$. To prove this decoupling theorem, we rewrite (3.12) as

$$\frac{f_{\mathrm{B}^*}}{f_\mathrm{B}} = 1 - C_F\frac{\alpha_\mathrm{s}^{[N_f]}(\mu)}{2\pi} + C_F\left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2\left[A + B^{[N_f]}(\mu)\right], \tag{3.14}$$
$$B^{[N_f]}(\mu) = \frac{44}{3}C_A\log\frac{m}{\mu} - \frac{16}{3}T_F\sum_{i=1}^{N_f}\left\{\log\frac{m}{\mu} - \frac{19}{12} + \Delta_2\left(\frac{m_i}{m}\right) - 2\Delta_3\left(\frac{m_i}{m}\right)\right\},$$

where A is a constant, specifying the quenched term in (3.12), and we have transformed the $\overline{\mathrm{MS}}$ coupling to an arbitrary scale μ, using the one–loop N_f–flavour β–function. The explicit μ–dependence in (3.14) at $\mathrm{O}(\alpha_\mathrm{s}^2)$ is cancelled by the implicit dependence at $\mathrm{O}(\alpha_\mathrm{s})$. Now, suppose that we take account of the existence of a super–heavy quark, with mass $m_h \gg m$. The effect is merely to replace N_f by N_f+1 in (3.14), where

$$\frac{\alpha_\mathrm{s}^{[N_f+1]}(\mu)}{2\pi} = \frac{\alpha_\mathrm{s}^{[N_f]}(\mu)}{2\pi} - \frac{16}{3}T_F\left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2\left\{\log\frac{m_h}{\mu}\right\} \tag{3.15}$$

ensures that $\alpha_\mathrm{s}^{[N_f+1]}(m_h) = \alpha_\mathrm{s}^{[N_f]}(m_h)$, and

$$B^{[N_f+1]}(\mu) = B^{[N_f]}(\mu) - \frac{16}{3}T_F\left\{\log\frac{m}{\mu} - \frac{19}{12} + \Delta_2\left(\frac{m_h}{m}\right) - 2\Delta_3\left(\frac{m_h}{m}\right)\right\} \tag{3.16}$$

includes the loop–effect of the super–heavy quark in diagram 1b. Referring to (A.5), we find that $\Delta_2(r) - 2\Delta_3(r) = \log r + \frac{19}{12} + \mathrm{O}(r^{-2}\log r)$, for $r = m_h/m \gg 1$. Hence the terms in braces in (3.15) and (3.16) cancel, at any scale μ. This also demonstrates that one must match the $\overline{\mathrm{MS}}$ couplings at $\mu = m_h$, as in (3.15), and not at, say, $\mu = 2m_h$. Interestingly, the external heavy flavour is also numerically unimportant. Its contribution to (3.12), relative to that of a massless quark, is suppressed by a factor

$$1 - \tfrac{12}{19}\left\{\Delta_2(1) - 2\Delta_3(1)\right\} = \tfrac{1}{19}(4\pi^2 - 41) = -0.08\,. \tag{3.17}$$

Comparable suppressions occur in all our results.

Turning to the numerical significance of our two–loop result, we substitute $N_l = 4$ in (3.12), and add the $1/m$ correction of [13, 14], obtaining

$$\frac{f_{\mathrm{B}^*}}{f_{\mathrm{B}}} = 1 - \frac{2}{3}\frac{\alpha_{\mathrm{s}}(m_b)}{\pi} - \left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2 (6.37 + \Delta_c) + \frac{1}{m_b}\left(\frac{2}{3}\bar{\Lambda} - 8G_2(m_b)\right) + \mathrm{O}(\alpha_{\mathrm{s}}^3, \alpha_{\mathrm{s}}/m_b, 1/m_b^2)\,, \tag{3.18}$$

with a finite c–quark mass correction $\Delta_c = \frac{2}{9}(\Delta_2(m_c/m_b) - 2\Delta_3(m_c/m_b)) = 0.18 \pm 0.01$, for a mass ratio $m_c/m_b = 0.28 \pm 0.03$, obtained from $m_b = (4.8 \pm 0.2)$ GeV, using $m_{\mathrm{B,D}} = m_{b,c} + \bar{\Lambda} + (\mu_\pi^2 - \mu_G^2)/(2m_{b,c})$ and $m_{\mathrm{B}^*} = m_{\mathrm{B}} + 2\mu_G^2/(3m_b)$ with a kinetic term $\mu_\pi^2 = (0.5 \pm 0.1)\ \mathrm{GeV}^2$. The current world average $\alpha_{\mathrm{s}}(m_Z) = 0.117 \pm 0.005$ [28], evolved down to m_b using the three–loop formula, gives $\alpha_{\mathrm{s}}(m_b) = 0.215 \pm 0.018$. Therefore the one–loop correction is $-(4.6 \pm 0.4)\%$, and the two–loop term is comparable to it: $-(3.1 \pm 0.5)\%$. Another way to state the slow convergence of the perturbation series is to say that the fastest–apparent–convergence scheme, with a vanishing two–loop correction in (3.14), would require one to evaluate $\alpha_{\mathrm{s}}(\mu)$ at far too low a scale: $\mu = m_b/13 = 370$ MeV.

The chromomagnetic interaction matrix element $G_2(m_b)$ is not well known; sum–rule estimates range between $G_2(m_b) = -26$ MeV [13], and $G_2(m_b) = +21$ MeV [14], with large uncertainties. Including the meson residual energy, $\bar{\Lambda}$, one obtains a $1/m$ correction of $+(11 \pm 3)\%$, from [13], and $+(4 \pm 3 \pm 2 \pm 2)\%$, from [14], whose three sources of uncertainty arise from: sum–rule fitting; an unknown two–loop anomalous dimension; and an uncertainty in $\bar{\Lambda}$. Combining our two–loop radiative corrections with this range of $1/m$ effects, we arrive at $f_{\mathrm{B}^*}/f_{\mathrm{B}} = 1.00 \pm 0.04$, with an uncertainty reflecting the difference between [13] and [14]. It is, however, quite unclear whether the perturbation series in α_{s} is converging fast enough for us to neglect the unknown three–loop term.

For the last ratio,

$$\frac{<0|(\bar{q}\gamma_i\gamma_0 Q)_\mu|\mathrm{B}>}{<0|\bar{q}\gamma_i Q|\mathrm{B}>} = \frac{f_{\mathrm{B}^*}^T(\mu)}{f_{\mathrm{B}^*}}\,, \tag{3.19}$$

we obtain

$$\begin{aligned}\frac{f_{\mathrm{B}^*}^T(m)}{f_{\mathrm{B}^*}} = \frac{C_{\gamma_0\gamma_1}(m)}{C_{\gamma_1}(m)} &= 1 \\ &+ C_F\left(\frac{\alpha_{\mathrm{s}}}{4\pi}\right)^2\left[C_F\frac{1}{3}\left(\frac{307}{8} - 70\zeta(2) + 8I\right) + C_A\frac{1}{3}\left(-\frac{4277}{72} + 24\zeta(2) - 4I\right)\right. \\ &\left.- \frac{8}{3}T_F\sum_{i=1}^{N_f}\left\{-\frac{205}{144} - \zeta(2) + \Delta_1\left(\frac{m_i}{m}\right) + \Delta_3\left(\frac{m_i}{m}\right)\right\}\right] \\ &\approx 1 - (4.69 - 0.34N_l + 0.04)\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)^2.\end{aligned} \tag{3.20}$$

The one–loop term vanishes at $\mu = m$, but will of course appear at any other μ. If one wants $f_{\mathrm{B}^*}^T(\mu)$ at a scale μ widely separated from m, one needs to use the QCD three–loop anomalous dimension of the current $\bar{q}\sigma_{\mu\nu}q$, in conjunction with (3.20). This anomalous dimension is not known at present, though it could be calculated using standard massless–quark methods.

Finally we note an intriguing pattern in the results (3.10), (3.12) and (3.20): in each a reasonable estimate of the two–loop term is obtained by the simple device of replacing N_l by $N_l - \frac{33}{2}$ in the easily computed light–quark–loop contributions. We call this device "naive non–abelianization", since it is based on the hope that results in the non–abelian theory may be estimated merely by replacing the leading term in the abelian large–N_f β–function by its non–abelian counterpart. Comparing naively non–abelianized estimates with our exact two–loop coefficients, we find that, with $N_l = 4$, the two–loop term in $m/\overline{m}(m)$ is overestimated by only 9% in (3.10), whilst in (3.12) one obtains an underestimate by 31%, and in (3.20) an overestimate by 27%. We thus see some merit in estimating radiative corrections by naive non–abelianization, in cases where a large–N_f calculation is practicable, whilst an exact one is not. It is noteworthy that naive non–abelianization of the large–N_f terms [26] in $\mathrm{e}^+\mathrm{e}^-$ annihilation underestimates the three–loop terms in R by less than 6%, with 4 or 5 active quark flavours.

4 All–order results

We now apply the methods of [29, 30, 31] to study the matching coefficients, at all orders of perturbation theory, in a highly fictitious limit: $N_f \to -\infty$.

As in other recent studies [15, 16, 17], the intention is to investigate so–called renormalon effects, associated with factorial growth of the coefficients of perturbation series [32]. Unfortunately, there is no gauge–invariant prescription that enables one to obtain all–order results in the asymptotically–free theory of interest; instead one merely studies a single chain of fermion loops, such as that which gives rise to the Landau pole in QED. The hope is that one may learn something about the non–abelian theory, by imagining that asymptotic freedom still holds with an infinite number of massless flavours. In practice, the analysis amounts to no more than changing the sign of N_f in the large–N_f methods of [29, 30, 31].

4.1 Master formula

Following recent practice, we disguise the sleight–of–hand, by hiding the large–N_f contribution to the β–function in a (fictitiously) positive value of $\beta_0 = \frac{1}{3}(11C_A - 4T_F N_f)$. In other words, we imagine that $\beta = -\mathrm{d}\log\alpha_\mathrm{s}/\mathrm{d}\log\mu = \beta_0\alpha_\mathrm{s}/2\pi + \mathrm{O}(\alpha_\mathrm{s}^2)$ remains finite and positive, so that the QCD behaviour $\beta \sim 1/\log(\mu/\Lambda_{\overline{\mathrm{MS}}})$ still holds at large μ.

In the matching coefficient we retain only the leading terms, of order $1/\beta_0$, in the limit $\beta_0 \to \infty$, whilst retaining all powers of $\beta \sim 1/\log(\mu/\Lambda_{\overline{\mathrm{MS}}})$. As the heavy–quark loop is negligible, in comparison with loops from a large number of massless quarks, we immediately obtain $\tilde{\Gamma}_0 = \Gamma_\mathrm{B}$ and $\tilde{Z}_Q^\mathrm{os} = 1$. A single L–loop diagram contributes to Γ_0. It is obtained by inserting a chain of $L-1$ massless–quark loops in the gluon line of diagram 1a. A corresponding insertion yields the L–loop diagram contributing to Z_Q^os. As $\beta_0 \to \infty$, the multiplications in (1.4) degenerate to mere addition and subtraction of terms of order $1/\beta_0$. Hence multiplicative renormalization of the QCD and HQET currents amounts to no more than minimal subtraction of powers of $1/\varepsilon$. Moreover, there is no mass counterterm to consider at $\mathrm{O}(1/\beta_0)$, and coupling–constant renormalization

amounts only to $\beta_0 g_0^2/(4\pi)^2 = \bar{\mu}^{2\varepsilon}\beta/(2+\beta/\varepsilon)$, with $\bar{\mu}^2 = \mu^2 \mathrm{e}^{\gamma}/4\pi$. Hence the entire perturbation series may be written, formally, as follows

$$C_\Gamma(\mu) = 1 + \sum_{L=1}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \left(\frac{\beta}{2\varepsilon+\beta}\right)^L - (\text{minimal subtractions}) + \mathrm{O}\left(\frac{1}{\beta_0^2}\right), \tag{4.1}$$

where $F(\varepsilon, u)$ is regular at $\varepsilon = u = 0$, and is easily calculated, in terms of a massless one–loop integral, in the gluon proper self–energy, and two massive on–shell one–loop integrals, determining the contributions of Z_Q^{os} and Γ_0.

As in the case of deep–inelastic sum rules at large N_f [26], the computational burden is slight, in comparison with the analysis of F_{32} hypergeometric functions in the two–loop integrals required for current correlators in QED [30] and HQET [15]. We find that

$$F(\varepsilon, u) = -\frac{C_F}{\beta_0}\left(\frac{\mu}{m}\right)^{2u} \frac{\Gamma(1+u)\Gamma(1-2u)}{\mathrm{e}^{-\gamma\varepsilon}\Gamma(3-u-\varepsilon)} \frac{N(\varepsilon,u)}{[D(\varepsilon)]^{1-u/\varepsilon}}, \tag{4.2}$$

$$N(\varepsilon,u) = (3-2\varepsilon)(1-u)(1+u-\varepsilon) + 2 - u - \varepsilon + 2\eta(n-2+\varepsilon)u - 2(n-2+\varepsilon)^2, \tag{4.3}$$

$$D(\varepsilon) = 6\mathrm{e}^{\gamma\varepsilon}\Gamma(1+\varepsilon)B(2-\varepsilon, 2-\varepsilon) = 1 + \tfrac{5}{3}\varepsilon + \mathrm{O}(\varepsilon^2). \tag{4.4}$$

The first term in (4.3) derives from Z_Q^{os}; the remaining terms, from Γ_0, are quadratic in $h = \eta(n-2+\varepsilon)$, since they result from an essentially one–loop calculation. The function (4.4) comes from the gluon self–energy.

We now follow the methods of [29, 30], expanding $F(\varepsilon, L\varepsilon)$ in powers of ε and $L\varepsilon$, and expanding $[\beta/(2\varepsilon+\beta)]^L$ in powers of $\beta/2\varepsilon$, to obtain a quadruple sum in (4.1). As shown in [29], combinatoric identities relate $1/\varepsilon$ terms, and hence $\overline{\mathrm{MS}}$ subtractions, to the Taylor coefficients of $F(\varepsilon, 0)$. In the case of the matching coefficient (1.4), we simply obtain

$$\tilde{\gamma}_J - \gamma_J = \beta F(-\beta/2, 0) + \mathrm{O}\left(\frac{1}{\beta_0^2}\right). \tag{4.5}$$

As shown in [30], the finite terms receive contributions from the Taylor coefficients of $F(\varepsilon, 0)$ and also from the Taylor coefficients of $F(0, u)$. The former are scheme–dependent, and give a well–behaved series in β. The latter are scheme–independent, and give a series that is not Borel–summable. A formal statement of the result may be written very simply:

$$C_\Gamma(\mu) = 1 + \int_{-\frac{\beta}{2}}^{0} \mathrm{d}\varepsilon \, \frac{F(0,0) - F(\varepsilon,0)}{\varepsilon} + \int_0^\infty \mathrm{d}u \exp\left(\frac{-2u}{\beta}\right) \frac{F(0,u) - F(0,0)}{u} + \mathrm{O}\left(\frac{1}{\beta_0^2}\right). \tag{4.6}$$

The function $F(0, u)$ specifies the Borel transform of the scheme–independent contributions [15, 16]. Attempting to undo the Borel transform, via the notional Laplace transform of (4.6), one encounters singularities at $u > 0$, which are referred to as infrared renormalons [32]. They reflect the fact that at higher and higher orders in perturbation theory one is probing regions of smaller and smaller gluon momenta. The first infrared renormalon occurs at $u = \frac{1}{2}$, giving a singularity in the integral with a residue that is a multiple of $\exp(-1/\beta)\mu/m \sim \Lambda_{\overline{\mathrm{MS}}}/m$. Any attempt to sum the perturbation series involves an arbitrary choice of how to deal with this singularity. In general, a singularity

at $u = u_0 > 0$ has a residue that is a multiple of $(\Lambda_{\overline{\rm MS}}/m)^{2u_0}$. Thus the perturbation series, taken to all orders, has ambiguities that are formally commensurate with the higher–dimension operators in the $1/m$ expansion (1.1).

The existence of such ambiguities is profoundly unsurprising. We see from (3.13) that the introduction of an infrared regulator, λ, such as a gluon mass, would modify the result of on–shell integrals at order λ/m. The infrared renormalon at $u = \frac{1}{2}$ serves to remind one of this simple fact. The situation is in close analogy with the operator–product expansion of QCD current correlators, at large $Q^2 = -q^2$, where the first infrared renormalon, at $u = 2$, leads to an ambiguity of order $(\Lambda_{\overline{\rm MS}}/Q)^4$, commensurate with the gluon–condensate contribution [32]. Thus one should not flinch at renormalons; they serve as healthy reminders that one has chosen to integrate over all gluon momenta, including the infrared region where non–perturbative effects are dominant. In fact, infrared renormalons have a positive virtue: the pattern of their residues must match the contributions of higher–dimension operators and thus provides consistency checks on the form of the $1/m$ expansion of HQET, or the $1/Q^2$ expansion of massless QCD. Since we have not taken the (inordinate) trouble of using an infrared regulator, the factorization of short– and long–distance physics into coefficients and operators is intrinsically ambiguous; each resummation prescription corresponds to a different set of values for the operator matrix elements. Nature, however, is not as slip–shod as we; she takes care to ensure that physical quantities are independent of the exigencies of our calculations.

Hence we see, from (4.5) and (4.6), that the simple multinomial (4.3) furnishes detailed information about anomalous dimensions, through its ε–dependence, and about renormalon singularities, through its u–dependence. Moreover, it does so for all current matchings, with n specifying the QCD current and $\eta = \pm 1$ the specific components that are matched to HQET currents. We now unpack some of this wealth of information.

4.2 Anomalous dimensions

Since we know that γ_J vanishes at $n = 1$, we can obtain both γ_J and $\tilde{\gamma}_J$ from (4.5):

$$\gamma_J = C_F \frac{\alpha_{\rm s}}{2\pi} \frac{(n-1)(d-1-n)}{18 B(d/2, d/2) B(d/2+1, 3-d/2)}\bigg|_{d=4+\beta}\,, \quad \tilde{\gamma}_J = \tfrac{1}{2}\gamma_J\Big|_{n=0}\,, \tag{4.7}$$

at order $1/\beta_0$. For the QCD currents, we verify the $n = 0$ result of [29], and the $n = 3$ analysis of [26]. The $n = 2$ and $n = 4$ results, as well as that for $\tilde{\gamma}_J$, are, we believe, new. It is rather intriguing that $\tilde{\gamma}_J - \frac{1}{2}\gamma_{\bar{q}q}$ vanishes at $\mathrm{O}(1/\beta_0)$, thereby continuing, at all orders, a trend already apparent in the simple N_f–independent two–loop formula given for this combination in [9]. A consequence of the all–order result is that the coefficient of the light–quark condensate, in the operator–product expansion of the HQET current correlator, is scale–independent, at leading order in $1/\beta_0$. This is confirmed, at two loops, by [27]. The all–order result may be somewhat accidental, since all large–N_f anomalous dimensions are fairly simply related. For example, we find, by similar methods, that the QCD field, q, and the HQET field, $\tilde{Q}$, have the following large–N_f off–shell anomalous dimensions:

$$\gamma_q = aC_F \frac{\alpha_{\rm s}}{\pi} - \tfrac{1}{2}\beta\tilde{\gamma}_J\,, \quad \tilde{\gamma}_Q = aC_F \frac{\alpha_{\rm s}}{\pi} + \tfrac{1}{2}(4+\beta)\tilde{\gamma}_J\,, \tag{4.8}$$

at order $1/\beta_0$. The dependence on the gauge parameter, a, is limited to the one–loop level. In the Landau gauge, with $a = 0$, the light–quark result agrees with [33] and the heavy–quark result with [15]. Impressively, all the $\mathrm{O}(1/\beta_0^2)$ terms in the anomalous dimension of the electron field were obtained in the QED analysis of [33].

We have a strong check of (4.8). The HQET current anomalous dimension is

$$\tilde{\gamma}_J = \tilde{\gamma}_\Gamma + \tfrac{1}{2}(\gamma_q + \tilde{\gamma}_Q)\,, \tag{4.9}$$

where $\tilde{\gamma}_\Gamma = \mathrm{d}\log\tilde{Z}_\Gamma/\mathrm{d}\log\mu$, $\tilde{Z}_\Gamma$ being the renormalization constant of the HQET proper–vertex function. It can be obtained from the $1/\varepsilon$ pole of the proper vertex at zero light–quark momentum and non–zero heavy–quark residual energy, because there are no infrared divergences in this case. At order $1/\beta_0$, all vertex diagrams with more than one loop contain a transverse gluon propagator, with quark–loop insertions. The argument in Section 2.1 shows that such diagrams vanish. Therefore, at $\mathrm{O}(1/\beta_0)$, there are no multi–loop contributions to $\tilde{\gamma}_\Gamma$. Moreover, the one–loop contribution vanishes in the Landau gauge. Taking account of this, the field anomalous dimensions (4.8) reproduce $\tilde{\gamma}_J$.

One easily obtains the perturbative expansions of (4.7) and (4.8), from the ε–expansion

$$\begin{aligned}\frac{1}{18B(d/2,d/2)B(d/2+1,3-d/2)} &= \frac{\sum_{k=0}^\infty\left(\frac{1}{2^k}-\frac{2k}{3}\right)\varepsilon^k}{\exp\left\{\sum_{s=3}^\infty\left(3+[-1]^s-2^s\right)\zeta(s)\varepsilon^s/s\right\}} \\ &= 1-\tfrac{1}{6}\varepsilon-\tfrac{13}{12}\varepsilon^2+\left(2\zeta(3)-\tfrac{15}{8}\right)\varepsilon^3+\left(3\zeta(4)-\tfrac{125}{48}\right)\varepsilon^4+\mathrm{O}\left(\varepsilon^5\right)\,,\end{aligned} \tag{4.10}$$

which reproduces the large–N_f features of all existing two– and three–loop calculations, and provides checks for future perturbative calculations. It is notable that $\zeta(4)$, which is conspicuously absent from three– and four–loop QCD results, is bound to occur at higher orders. Whilst the perturbative expansion (4.10) is valid only for $|\varepsilon| < \frac{1}{2}$, corresponding to $|\beta| < 1$, one can see from the singularities of the Euler–Beta functions in (4.10) that in fact no singularity is encountered for $d > -1$. Hence the anomalous dimensions are well defined for $\beta > -5$, and summation of the large–N_f perturbation series results in an extension of the domain of convergence.

4.3 Renormalon ambiguities and their cancellation

In stark contrast to the first integral in (4.6), which exists for all $\beta > -5$, the second contains infrared renormalons, for all $\beta > 0$. If one arbitrarily chose to evaluate it by principal–value integration, one would differ, at order $\Lambda_{\overline{\mathrm{MS}}}/m$, with someone who chose to add some (presumably real) multiple of the residue at $u = \frac{1}{2}$. A solution would be to absorb this disagreement into different values adopted for the other terms that occur at $\mathrm{O}(1/m)$ in (1.1).

The $1/m$ corrections to the ground–state meson matrix elements were calculated in [13] for the currents with $n = 1$. We performed the corresponding calculation for an arbitrary Γ, obtaining a general result in terms of $h|_{\varepsilon=0} = \eta(n-2)$:

$$\begin{aligned}&\frac{<0|J(\mu)|\mathrm{M}>}{<0|\tilde{J}(\mu)|\mathrm{M}>} = C_\Gamma(\mu) + \frac{1}{m}\left(G_1 + 2d_\Gamma G_2 - \frac{1}{6}\bar{d}_\Gamma(m_\mathrm{M}-m)\right) + \mathrm{O}\left(\frac{1}{m^2}\right)\,, \\ &\bar{d}_\Gamma = 1-2\eta(n-2)\,,\quad d_\Gamma = \tfrac{1}{2}\left(\bar{d}_\Gamma^2-3\right)\,,\end{aligned} \tag{4.11}$$

where M is the ground–state or excited–state meson that couples to the current, and G_1 and G_2 are bilocal matrix elements of the heavy–quark kinetic energy and chromomagnetic interaction, respectively. Multiplying Γ by γ_0 changes the sign of $\bar{d}_\Gamma$, without changing d_Γ, since the latter is obtained from $\sigma_{\mu\nu}\Gamma\frac{1+\gamma_0}{2}\sigma^{\mu\nu}\frac{1+\gamma_0}{2} = 2d_\Gamma\Gamma\frac{1+\gamma_0}{2}$. We find that: $\bar{d}_\Gamma = (-1)^{n+1}d_\Gamma$ for all 8 currents; $d_\Gamma = 3$ for the 4 currents that couple to $0^\pm$ mesons; $d_\Gamma = -1$ for those that couple to $1^\pm$ mesons. The ground–state S–wave 0^- and 1^- mesons remain degenerate at this order in $1/m$, as do the excited–state P–wave 0^+ and 1^+ mesons.

Viewed from the standpoint of HQET, the residual–energy term, $\bar{\Lambda} = m_{\mathrm{B}} - m_b + \mathrm{O}(1/m_b)$, and the matrix elements, $G_{1,2}$, suffer from ultraviolet–renormalon ambiguities [15]. HQET knows nothing about the pole mass, m_b, since it deals only with residual energies. However, the HQET self–energy, $\tilde{\Sigma}(\omega)$, has a linear ultraviolet divergence, conventionally suppressed by dimensional regularization. If one were to use an ultraviolet momentum–space cut–off, it would be necessary to introduce a residual mass, into which this linear divergence could be absorbed. Moreover, $G_{1,2}$ are bound to acquire ultraviolet–renormalon ambiguities, via mixing with $\bar{\Lambda}$ [17]. We now determine these by demanding that they cancel the infrared–renormalon ambiguities in (4.11).

Setting $\varepsilon = 0$ in the master result (4.2) and (4.3), we obtain

$$F(0,u) = -\frac{C_F}{\beta_0}\left(\frac{\mu \mathrm{e}^{\frac{5}{6}}}{m}\right)^{2u}\frac{B(1+u,1-2u)}{2-u}\left[5-u-3u^2+2\eta(n-2)u-2(n-2)^2\right]. \tag{4.12}$$

Comparing the residue at $u = \frac{1}{2}$ with that for the pole mass [15, 16], we obtain the ambiguity of the matching coefficients due to the infrared renormalon at $u = \frac{1}{2}$, in terms of the pole–mass ambiguity, Δm:

$$\Delta C_\Gamma(\mu) = -\frac{1}{3}\left[\frac{15}{4}+\eta(n-2)-2(n-2)^2\right]\frac{\Delta m}{m}\,, \tag{4.13}$$

at any scale, μ. This ambiguity must be compensated in (4.11) by ultraviolet–renormalon ambiguities ΔG_1 and ΔG_2 [17]. Since (4.13) is quadratic in $\eta(n-2)$, we have 3 equations, with only 2 unknowns. They are indeed consistent, and yield

$$\Delta G_1 = \tfrac{3}{4}\Delta m\,, \quad \Delta G_2 = -\tfrac{1}{6}\Delta m\,. \tag{4.14}$$

This result was obtained in [17], where the matching coefficients of the $\eta = \pm 1$ components of the $n = 1$ current were considered. However, the 2 ambiguities (4.14) were obtained in [17] from only 2 equations, with no consistency check. We have one check resulting from one extra equation.

Finally, one might wonder what are the prospects of going to $\mathrm{O}(1/\beta_0^2)$, where the renormalon structure is presumably far less trivial, with the possibility of cuts appearing in the Borel transform, rather than mere poles. The prospects appear to be rather slim, since one would have to insert chains of light–quark loops into the gluon lines of the diagrams of Fig. 1. Even worse, one would need to insert a loop into the three–gluon vertices. There is, however, one term that is clearly tractable: that obtained by replacing the one–fermion–loop boson self–energy of diagram 1b by self–energy terms with L loops, of which $L-2$ are fermion loops. This was achieved in an $\mathrm{O}(1/N_f^2)$ analysis of the muon anomaly [30], where the large–L behaviour was obtained, by evaluating an integral of

the Borel transform of the corresponding contributions to the Gell–Mann—Low function, also obtained in closed form in [30]. Recently, corresponding contributions to the HQET self–energy have been analyzed [34], with a result identical to that in [30].

5 Summary

Our principal result is formula (2.19), which matches QCD and HQET currents in (1.3). It applies to all currents with a Dirac structure $\Gamma = \gamma^{[\mu_1} \ldots \gamma^{\mu_n]}$ that anticommutes or commutes with γ_0. The one–loop term agrees with [1]. The coefficients in (2.20) and (2.21) give our new two–loop term, for any gauge theory, current, and quark–mass ratios. To use the general result one sets n to the number of γ–matrices in Γ and uses $\eta = +(-1)^n$, or $\eta = -(-1)^n$, according as whether Γ anticommutes or commutes with γ_0. Multiplying Γ by a naively anticommuting γ_5 does not change the matching coefficient. Table 1 gives specific numerical results, ignoring small effects of light/heavy quark–mass ratios.

All–order results, of similar generality, were obtained from the master formula (4.2), which gives the $\mathrm{O}(1/\beta_0)$ terms in the matching coefficients (4.6) and the anomalous dimensions (4.7). Infrared–renormalon ambiguities in the matching coefficients are cancelled by the ultraviolet–renormalon ambiguities (4.14), derived in [17] without benefit of our consistency check.

We briefly note the following salient points.

1. Consistent results were obtained by including heavy–quark–loop effects in all 6 of the terms of (1.4); one must not omit heavy loops from HQET.

2. The HQET on–shell renormalization constant (2.4) removes the non–uniformity, at zero light–quark mass, of the corresponding QCD result [12], as shown in (2.5).

3. The current–independent procedure of (2.8) and (2.9) is an efficient alternative to evaluating traces with complicated antisymmetrized products of γ–matrices.

4. Effects of non–trivial quark–mass ratios are obtained from the 3 integrals defined and evaluated in Appendix A, where series expansions are given.

5. Apart from such two–scale integrals, our calculations were purely algebraic, as exemplified by the polynomial coefficients of Appendix B for the QCD vertex (2.13).

6. All two–loop anomalous dimensions of QCD currents are given by (2.17), whose specialization (2.18) to the tensor current appears to be new.

7. The matching coefficients (2.19) confirm the universality of the finite renormalizations (3.3) and (3.5) that restore chiral symmetry to the 't Hooft–Veltman γ_5 [23, 24].

8. Two–loop corrections to ratios of meson decay constants are given: in (3.10), which confirms the relation between $\overline{\mathrm{MS}}$ and pole masses [11]; in (3.12), which gives a $-(3.1 \pm 0.5)\%$ two–loop correction to $f_{\mathrm{B}^*}/f_{\mathrm{B}}$, comparable to the $-(4.6 \pm 0.4)\%$ one–loop correction; and in (3.20), which gives the tensor coupling of B^*.

9. All our two–loop corrections are distressingly large; fastest apparent convergence of $f_{\mathrm{B}^*}/f_{\mathrm{B}}$ would require evaluating α_{s} at $\mu = 370$ MeV.

10. Infrared safety of ratios of matching coefficients, at finite orders of perturbation theory, is exemplified by (3.13), which also exposes the infrared renormalon at $u = \frac{1}{2}$.

11. Decoupling of the t–quark loop from $f_{\mathrm{B}^*}/f_{\mathrm{B}}$ is explicitly demonstrated in (3.14).

12. The b–quark loop approximately decouples from $f_{\mathrm{B}^*}/f_{\mathrm{B}}$, as shown by (3.17).

13. Naive non–abelianization of the massless–quark–loop contributions, by the process $N_l \to N_l - \frac{33}{2}$, approximates our exact two–loop terms, in ratios of decay constants, at the 30% level, or better.

14. In the $\overline{\mathrm{MS}}$ scheme, every large–N_f series of the type (4.1) is formally resummed by (4.6), whose first integral gives the scale–dependence, whilst the second contains scheme–independent renormalon singularities.

15. Anomalous dimensions and renormalon residues, for any current, are encoded by the ε– and u–dependencies of the master multinomial (4.3).

16. The anomalous current dimensions (4.7) involve the universal [15, 26, 29, 30, 33] ε–expansion (4.10).

17. The anomalous field dimensions (4.8) confirm our result $\tilde{\gamma}_J = \frac{1}{2}\gamma_{\bar{q}q} + \mathrm{O}(1/\beta_0^2)$.

18. The $\mathrm{O}(1/m)$ contributions for any current matching are given by (4.11).

19. The residue in (4.12) at $u = \frac{1}{2}$ produces the ambiguity (4.13), which is cancelled in (4.11) by ambiguities in the pole mass, m, and the matrix elements $G_{1,2}$.

20. The inclusion of $\mathrm{O}(1/\beta_0^2)$ terms in all–order calculations will be very demanding, though some progress in the abelian case has been made in [30, 33, 34].

In conclusion: we hope that our results will serve both to enable more accurate extraction of meson decay constants from lattice simulations and sum rules, and also to provide detailed testing grounds for perturbative and non–perturbative approximations based on large–N_f expansions.

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A Dilogarithmic integrals $\Delta_k(r)$

The finite–mass corrections (2.6) and (2.16) involve 3 integrals of the polarization operator subtracted at $m_i = 0$. With $z = -m_i^2/k^2$, this takes the form [11]

$$\Pi(z) = 2(1-2z)\sqrt{1+4z}\,\mathrm{arccoth}\,\sqrt{1+4z} + \log z + 4z\,, \tag{A.1}$$

which is then integrated over simple functions of $y = 2/(1+\sqrt{1-4m^2/k^2})$, to obtain

$$\begin{aligned}
\Delta_1(r) &= \frac{1}{6}\int_0^1 \mathrm{d}y \frac{4-y}{1-y}\Pi\left(r^2\frac{1-y}{y^2}\right) = -(1+r)L_+(r) - (1-r)L_-(r) + \log^2 r + \zeta(2),\\
\Delta_2(r) &= \frac{2}{3}\int_0^1 \mathrm{d}y \Pi\left(r^2\frac{1-y}{y^2}\right) = -r(1-r^2)L_+(r) + r(1-r^2)L_-(r) + 2r^2(\log r + 1),\\
\Delta_3(r) &= \frac{1}{6}\int_0^1 \mathrm{d}y\, y\Pi\left(r^2\frac{1-y}{y^2}\right) = -r^3(1+r)L_+(r) + r^3(1-r)L_-(r) - r^2\left(\log r + \frac{3}{2}\right),
\end{aligned} \tag{A.2}$$

in terms of the dilogarithmic integrals [11, 12]

$$L_\pm(r) = \int_0^1 \mathrm{d}x \frac{\log x - \log r}{x \pm r} = \begin{cases} \frac{1}{2}\log^2 r - \log r \log(1\pm r) + \frac{1\mp 3}{2}\zeta(2) - \mathrm{Li}_2(\mp r), & r \le 1\,,\\ \log r \log \frac{r}{r\pm 1} + \mathrm{Li}_2(\mp 1/r), & r \ge 1\,, \end{cases} \tag{A.3}$$

where $\mathrm{Li}_p(x) = \sum_{n=1}^\infty x^n/n^p$. Expanding (A.2), for $r < 1$, we obtain

$$\Delta_k(r) = 2\log r \sum_{n=1}^\infty g_k(n) r^{2n} + \sum_{n=1}^\infty g_k'(n) r^{2n} + \delta_k\,, \tag{A.4}$$

$$\begin{aligned}
g_1(n) &= \frac{1}{2n-1} - \frac{1}{2n}\,, \quad \delta_1 = 3\zeta(2)r\,,\\
g_2(n) &= \frac{1}{2n-1} - \frac{1}{2n-3}\,, \quad \delta_2 = 3\zeta(2)r(1-r^2)\,,\\
g_3(n) &= \frac{1}{2n-3} - \frac{1}{2n-4} \quad (n\neq 2)\,, \quad g_3(2) = 1\,, \quad g_3'(2) = -2\,,\\
&\delta_3 = -r^4\log^2 r + 3\zeta(2)r^3\left(1 - \tfrac{1}{3}r\right)\,,
\end{aligned}$$

where $g_k'(n) = \mathrm{d}g_k(n)/\mathrm{d}n$. Similarly, for $r > 1$, we find

$$\Delta_k(r) = -2\log r \sum_{n=0}^\infty \bar{g}_k(n)\frac{1}{r^{2n}} + \sum_{n=0}^\infty \bar{g}_k'(n)\frac{1}{r^{2n}} + \bar{\delta}_k\,, \tag{A.5}$$

$$\begin{aligned}
\bar{g}_1(n) &= \frac{1}{2n} - \frac{1}{2n+1} \quad (n\neq 0)\,, \quad \bar{g}_1(0) = -1\,, \quad \bar{g}_1'(0) = 2\,, \quad \bar{\delta}_1 = \log^2 r + \zeta(2)\,,\\
\bar{g}_2(n) &= \frac{1}{2n+3} - \frac{1}{2n+1}\,, \quad \bar{\delta}_2 = 0\,,\\
\bar{g}_3(n) &= \frac{1}{2n+4} - \frac{1}{2n+3}\,, \quad \bar{\delta}_3 = 0.
\end{aligned}$$

Finally, external–flavour contributions, with $m_i = m$, are obtained from

$$\Delta_1(1) = 2\zeta(2)\,, \quad \Delta_2(1) = 2\,, \quad \Delta_3(1) = \zeta(2) - \tfrac{3}{2}\,. \tag{A.6}$$

B QCD proper vertex

Here we present all non–zero coefficients in the formula (2.13) for the QCD proper vertex. The LaTeX source of these formulae has been generated by a REDUCE program, using the library package RLFI by R. Liska [20].

$$a_{000} = \left(3d^3 - 30d^2 + 101d - 110\right)(d-1)(d-2)(d-5)(d-6)$$
$$a_{001} = -2\left(d^5 - 17d^4 + 109d^3 - 325d^2 + 428d - 172\right)(d-5)(d-6)$$
$$a_{002} = -2\left(5d^4 - 50d^3 + 164d^2 - 156d - 56\right)(d-5)(d-6)$$
$$a_{003} = 8(d-4)^2(d-5)(d-6)$$
$$a_{010} = \left(d^2 - 8d + 10\right)(2d-7)(d-2)(d-4)(d-5)(d-6)$$
$$a_{011} = -2\left(d^3 - 13d^2 + 42d - 32\right)(d-4)^2(d-5)(d-6)$$
$$a_{012} = -2\left(3d^2 - 10d - 4\right)(d-4)^2(d-5)(d-6)$$
$$a_{013} = 8(d-4)^2(d-5)(d-6)$$
$$a_{030} = -2\left(7d^3 - 76d^2 + 255d - 246\right)(d-2)(d-3)(d-4)$$
$$a_{031} = 4\left(3d^4 - 50d^3 + 287d^2 - 660d + 492\right)(d-3)(d-4)$$
$$a_{032} = 32\left(d^2 - 6d + 6\right)(d-3)(d-4)^2$$
$$a_{100} = -\left(d^4 - 9d^3 + 26d^2 - 24d + 2\right)(3d-8)(d+2)$$
$$a_{101} = 4\left(2d^6 - 31d^5 + 183d^4 - 508d^3 + 642d^2 - 260d - 32\right)$$
$$a_{102} = 4\left(2d^5 - 2d^4 - 139d^3 + 794d^2 - 1600d + 1088\right)$$
$$a_{103} = -16(2d-7)(d-2)(d-4)$$
$$a_{110} = -\left(2d^4 - 24d^3 + 105d^2 - 199d + 134\right)(3d-8)(d-2)$$
$$a_{111} = 2\left(2d^5 - 33d^4 + 210d^3 - 643d^2 + 916d - 448\right)(d-4)$$
$$a_{112} = 2\left(8d^4 - 71d^3 + 211d^2 - 178d - 88\right)(d-4)$$
$$a_{113} = -8(2d-7)(d-3)(d-4)$$
$$a_{120} = 4(3d-8)(d-2)(d-3)(d-4)$$
$$a_{121} = -16(d-2)(d-3)(d-4)^2$$
$$a_{122} = -16(d-2)(d-3)(d-4)$$
$$a_{200} = -\left(d^3 - 12d^2 + 45d - 46\right)(3d-8)(d-6)$$
$$a_{201} = 2\left(d^3 - 18d^2 + 101d - 184\right)(d-2)(d-6)$$
$$a_{202} = 8\left(d^3 - 12d^2 + 47d - 56\right)(d-6)$$
$$a_{230} = 2\left(d^2 - 5d + 2\right)(3d-8)(d-4)$$
$$a_{231} = -4\left(d^3 - 11d^2 + 30d - 16\right)(d-4)$$
$$a_{232} = -16(d-2)(d-4)^2$$

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